Department of Mathematics and Statistics
Indian Institute of Technology Kanpur
End-Course Examination (September 19, 2013) Grading Scheme
MSO202A/MSO202: Introduction To Complex Analysis

Roll No.: ………………. Section: …………. **Time: 120 Minutes**
Marks: 120

Name: ……………………………………………. Seat Number…………

Note: Give only answers (no details of workout) for Problems 1 to 16 at dotted lines. Problems 1-8 are of 7 marks each and Problems 9-16 are of 8 marks each.

1. Let the circle $|z-z_0|=r$ pass through complex numbers α and $1/\bar{\alpha}$. If $|z_0|^2 \neq 1+r^2$, then

$|\alpha| = \ldots\ldots\ldots$.

Solution: *The given circle passes through* α *and* $1/\bar{\alpha}$ *implies*

$$|\alpha - z_0|^2 = r^2 \Rightarrow |\alpha|^2 + |z_0|^2 - 2\,\mathrm{Re}(\bar{\alpha} z_0) = r^2$$

$$|1-\bar{\alpha} z_0|^2 = |\alpha|^2 r^2 \Rightarrow 1 + |\alpha|^2 |z_0|^2 - 2\,\mathrm{Re}(\bar{\alpha} z_0) = |\alpha|^2 r^2$$

Subtracting the second equation from the first gives $(1-|\alpha|^2)(1+r^2-|z_0|^2)=0$. *Therefore,* $|z_0|^2 \neq 1+r^2 \Rightarrow |\alpha| = 1.$ ***(7 marks)***

2. Let $S = \{z : \mathrm{Im}(\frac{z-1-\sqrt{3}\,i)}{1+\sqrt{3}\,i}) < 0\} \cap \{z : |z| < 2\} \cap \{z : \mathrm{Re}\, z > 0\}$. Then,

Area of set $S = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Solution: *The region* S *is the sector of the disk* $|z|<2$ *lying in the right half plane, below the line* $y=\sqrt{3}\,x$. *This sector subtends an angle* $\frac{\pi}{3}+\frac{\pi}{2}=\frac{5\pi}{6}$ *at its center. Therefore, Area of* $S = \frac{1}{2} r^2 \theta = \frac{1}{2} \times 4 \times \frac{5\pi}{6} = \frac{5\pi}{3}$. ***(7 marks)***

3. The value of $\oint_\Gamma |z|^2\, \bar{z}\, dz$, where Γ is counter-clockwise oriented boundary of the region $D = \{z = x + i\, y : x^2 + 9y^2 < 1, y > 0\}$, is

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Solution: The boundary of D consists of the counter-clockwise oriented part of ellipse $C:\ z(t) = \cos t + i\frac{\sin t}{3}$ *in the upper half plane and the line segment* $[-1, 1]$. *Therefore,* $0 < t < \pi$. $\Rightarrow \int_\Gamma |z|^2\, \bar{z}\, dz = \int_C |z|^2\, \bar{z}\, dz + \int_{-1}^{1} x^3\, dx = I_1 + I_2$.

$I_2 = 0$ *and*

$$I_1 = \int_0^\pi (\cos^2 t + \frac{\sin^2 t}{9})(\cos t - i\frac{\sin t}{3})(-\sin t + i\frac{\cos t}{3})\ dt$$

$$=\frac{1}{81}\int_0^{\pi}(5+4\cos 2t)\ (-4\sin 2t+3i)dt=\frac{1}{81}\int_0^{\pi}(-8\sin 4t-20\sin 2t)+\frac{i}{27}\int_0^{\pi}\ (5+4\cos 2t)dt \qquad (*)$$

$$=0+\frac{i}{27}[2\sin 2t+5t]_0^{\pi}=\frac{5\pi i}{27}$$

(7 marks, deduct 2 marks if answer is left at step (*))

4. Under the mapping $w=\sin z$, the rectangular region $\{(x,y):0\le x\le\pi,\ 0\le y\le 3\}$ is mapped onto a region D. Then,

   $Area\ of\ D=$..................

Solution: *The function* $w=\sin z=\sin x\cosh y+i\cos x\sinh y$ *maps the given rectangular region in half-elliptical region D lying in right half plane with boundaries consisting of the line segments* $0\le u\le\cosh 3,\ -\sinh 3\le v\le\sinh 3$ *and the part of ellipse* $\frac{u^2}{\cosh^2 3}+\frac{v^2}{\sinh^2 3}=1$ *lying in right-half plane. Therefore, Area of D* $=\frac{\pi\cosh 3\sinh 3}{2}$.

(7 marks)

5. The expression of $\cot^{-1} z$ in terms of complex logarithmic function is

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Solution:

$$w=\cot^{-1}z\Rightarrow\cot w=z\Rightarrow\frac{i(e^{iw}+e^{-iw})}{(e^{iw}-e^{-iw})}=z\Rightarrow e^{2iw}=\log(\frac{z+i}{z-i})\Rightarrow w=\frac{i}{2}\log(\frac{z+i}{z-i})$$ ***(7 marks)***

6. Let $e^{e^{3z}}$ be bounded on the line $C_p=\{(x,y):-\infty<x<\infty,\ y=p\}$, where p is a constant. If p lies in any one of the intervals $[\alpha_k,\ \beta_k],\ k=0,\pm1,\pm2,.....$, then

   $(i)\ \alpha_k=$.......................... $(ii)\ \beta_k=$.......................... .

Solution: *Since* $\left|e^{e^{3z}}\right|=e^{e^{3x}\cos 3y}$ *and* $\cos 3y\le 0$ *for* $3y=3p\in[\frac{\pi}{2}+2k\pi,\ \frac{3\pi}{2}+2k\pi],\ k=0,\pm1,\pm2,.....,$ *the function* $e^{e^{3z}}$ *is bounded on line* C_p *for* $p\in[\frac{\pi}{6}+\frac{2k\pi}{3},\ \frac{\pi}{2}+\frac{2k\pi}{3}]$*. This gives* $\alpha_k=\frac{\pi}{6}+\frac{2k\pi}{3}$ *and* $\beta_k=\frac{\pi}{2}+\frac{2k\pi}{3}$.

(7 marks)

7. Let the function $g(z)=u(x,y)+i\,v(x,y)$ be defined by $g(z)=\dfrac{2z^{11}}{|z|^{10}}, if\ \ z\neq 0$ and $g(z)=0, if\ \ z=0$. Then,

(i) $u_x(0,0)=$ .... (ii) $u_y(0,0)=$ .... (iii) $v_x(0,0)=$ .... (iv) $v_y(0,0)=$ ....

Solution:

$$g(\Delta x,0)=\frac{2\Delta x^{11}}{|\Delta x|^{10}} \ and\ \ g(0,\Delta y)=\frac{-2i\Delta y^{11}}{|\Delta y|^{10}} \Rightarrow \frac{u(\Delta x,0)}{\Delta x}=\frac{2\Delta x^{10}}{|\Delta x|^{10}},\ \ \frac{v(\Delta x,0)}{\Delta x}=0,\ \ \frac{u(0,\Delta y)}{\Delta y}=0,\ \frac{v(0,\Delta y)}{\Delta y}=-\frac{2\Delta y^{10}}{|\Delta y|^{10}}$$

$\Rightarrow u_x(0,0)=2,\ u_y(0,0)=0,\ v_x(0,0)=0,\ \ v_y(0,0)=-2$.

(7 marks, 4 marks for (i) and (iv) correct, 3 marks for (ii) and (iii) correct)

8. Let the Taylor series of $h(z)=\dfrac{1}{1-z-z^2}$ around the point $z=0$ be $\sum_{n=0}^{\infty} a_n z^n$. If $\alpha,\beta\ (\alpha>\beta)$ are the roots of the equation $z^2+z-1=0$, then a_n in terms of α,β is

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Solution: $h(z)=\dfrac{1}{\beta-\alpha}[\dfrac{1}{\beta}(\dfrac{1}{1-z/\beta})-\dfrac{1}{\alpha}(\dfrac{1}{1-z/\alpha})],\quad \alpha=\dfrac{\sqrt{5}-1}{2},\ \beta=-\dfrac{\sqrt{5}+1}{2}$

$$\Rightarrow h(z)=\frac{1}{\beta-\alpha}\left[\sum_{n=0}^{\infty}\frac{z^n}{\beta^{n+1}}-\sum_{n=0}^{\infty}\frac{z^n}{\alpha^{n+1}}\right]=\frac{1}{\sqrt{5}}\sum_{n=0}^{\infty}\frac{\beta^{n+1}-\alpha^{n+1}}{(\alpha\beta)^{n+1}}z^n,\ |z|<\alpha$$

$\Rightarrow a_n=\dfrac{1}{\sqrt{5}}[\dfrac{\beta^{n+1}-\alpha^{n+1}}{(\alpha\beta)^{n+1}}]$.

(7 marks, deduct only 2 marks if answer differs by a negative sign)

9. If the largest annulus in which Laurent series $\sum_{n=1}^{\infty}\dfrac{3^{n^2}}{(2n)!}z^{n^2}+\sum_{n=1}^{\infty}4^{n^2}z^{-n^2}$ converges is $r_1<|z|<r_2$, then

$(i)\ r_1=$ .... $(ii)\ r_2=$ ....

Solution: *The series* $\sum_{n=1}^{\infty}\dfrac{3^{n^2}}{(2n)!}z^{n^2}$ *converges in* $|z|<\dfrac{1}{3}$, *since*

$$\lim_{n\to\infty}\left|\frac{a_{n-1}}{a_n}\right|^{1/(\lambda_n-\lambda_{n-1})}=\lim_{n\to\infty}\left|\frac{3^{(n-1)^2}}{(2n-2)!}\frac{(2n)!}{3^{n^2}}\right|^{1/\left(n^2-(n-1)^2\right)}=\lim_{n\to\infty}\left|\frac{(2n)(2n-1)}{3^{2n-1}}\right|^{1/(2n-1)}=\frac{1}{3}.$$

The series $\sum_{n=1}^{\infty}4^{n^2}z^{-n^2}$ *converges in* $|z|>4$, *since radius of convergence of series* $\sum_{n=1}^{\infty}4^{n^2}w^{n^2}$ *is* $\dfrac{1}{4}$, *with* $w=\dfrac{1}{z}$.

Therefore, largest annulus of convergence of given Laurent series is empty $\Rightarrow$ *such* $r_1,\ r_2$ *do not exist*.

(8 marks)

10. The residues of the function $\psi(z)=\dfrac{z^{101}}{(2z-1)^{100}}$ at its singularities in extended complex plane are

(i)………………… (ii)…………………

Solution: *The function* $\psi(z)$ *has two singularities at* $z=\dfrac{1}{2}$ *and* $z=\infty$ *in extended complex plane. For residue of* $\psi(z)$ *at* $z=\infty$ *, observe that* $\psi(1/z)=\dfrac{1/z^{101}}{(\frac{2}{z}-1)^{100}}=\dfrac{1}{z(2-z)^{100}}=\dfrac{1}{2^{100}z}(1-\dfrac{z}{2})^{-100}$

$$\Rightarrow -\text{'coefficient of } z \text{ in the Laurent's expansion of } \psi(1/z) \text{ around } 0\text{'}=-\frac{100\times 101}{2^{103}}$$

$$\Rightarrow \operatorname*{Res}_{z=\infty}\frac{z^{101}}{(2z-1)^{100}}=-\frac{100\times 101}{2^{103}}$$

$\operatorname*{Res}_{z=1/2}\dfrac{z^{101}}{(2z-1)^{100}}=\dfrac{100\times 101}{2^{103}}$ *(since sum of residues of a function in extended complex plane is zero)*

(8 marks, 4 marks for each correct answer)

11. Let γ be a simple, closed, piecewise smooth curve oriented counterclockwise and not passing through the points $z_1=5$ *and* $z_2=2$. Then, the possible values of $I=\oint_\gamma \dfrac{1}{(z^2-7z+10)^3}dz$ are

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Solution:

(i) None of the points $5,2$ *are enclosed by* γ *:* $I=0$ *(by Cauchy Theorem)*

(ii) The point 5 *is enclosed but the point* 2 *is not enclosed by* γ *:*

$I=\int_\gamma \dfrac{1}{(z-5)^3(z-2)^3}dz=\dfrac{2\pi i}{2!}\times[\dfrac{d^2}{dz^2}\dfrac{1}{(z-2)^3}]_{z=5}=\dfrac{2\pi i}{2!}\times 3\times 4\times\dfrac{1}{3^5}=\dfrac{4}{3^4}\pi i$ *(by Cauchy Integral Formula For derivatives)*

(iii) The point 2 *is enclosed but the point* 5 *is not enclosed by* γ *:*

$I=\int_\gamma \dfrac{1}{(z-5)^3(z-2)^3}dz=\dfrac{2\pi i}{2!}\times[\dfrac{d^2}{dz^2}\dfrac{1}{(z-5)^3}]_{z=2}=-\dfrac{2\pi i}{2!}\times 3\times 4\times\dfrac{1}{3^5}=-\dfrac{4}{3^4}\pi i$ *(by Cauchy Integral Formula For derivatives)*

(iv) Both of the points $2,5$ *are enclosed by* γ *:* $I=\int_\gamma \dfrac{1}{(z-5)^3(z-2)^3}dz=\dfrac{4}{3^4}\pi i-\dfrac{4}{3^4}\pi i=0$.

(8 marks, 5 marks for any two correct answers, 3 marks for any one correct answer)

12. Let $I = \int_{-\infty}^{\infty} \frac{e^{i\alpha x}}{\left(x^2+a^2\right)(x-ia)}\, dx$; $\alpha > 0,\ a > 0$. Then, the principal value of integral I is

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Solution: *The function* $f(z) = \frac{1}{\left(z^2+a^2\right)(z-ia)} \to 0$ *as* $z \to \infty$ *in the upper half plane and it has a pole of order* 2 *at* $z = ia$ *in the upper half plane.*

$$\therefore I = 2\pi i \operatorname*{res}_{z=ia}\left[\frac{e^{i\alpha z}}{\left(z^2+a^2\right)(z-ia)}\right] = 2\pi i\left\{\frac{d}{dz}\left(\frac{e^{i\alpha z}}{z+ia}\right)\right\}_{z=ia}$$

$$= 2\pi i\left[\frac{i\alpha e^{i\alpha z}(z+ia) - e^{i\alpha z}}{(z+ia)^2}\right]_{z=ia} = 2\pi i e^{-a\alpha}\left[\frac{1+2a\alpha}{4a^2}\right]$$

(8 marks, deduct 2 marks only for minor slips in answer)

13. Let $h(z) = \frac{\sin^{14} z}{z^3(z-1)^4}$. Then, change in the argument of $h(z)$ as z describes the circle $z(t) = 2e^{it},\ 0 \le t < 2\pi$, once in counterclockwise direction is

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Solution: *The function* $h(z)$ *has a zero of order* 11 *at* $z = 0$ *and a pole of order* 4 *at* $z = 1$*, lying in* $|z| < 2$ *and* $h(z) \ne 0$ *on* $|z| = 2$ *. Therefore, change in the argument of* $h(z)$ *as* z *describes the circle* $z(t) = 2e^{it},\ 0 \le t < 2\pi$*, once in counterclockwise direction is* $2\pi(11-4) = 14\pi$ *.*

(8 marks, deduct only 2 marks if answer is 7, deduct 2 marks if answer differs by a negative sign)

14. The number of zeros of the polynomial $P(z) = z^5 - z + 16$ lying in the annulus $1 < |z| < 2$ are

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Solution: *Let* $f_1(z) = z^5$ *and* $g_1(z) = -z + 16$ *. Then, on* $|z| = 2$ *,* $|f_1(z)| = 32$ *and* $|g_1(z)| < 18$
$\Rightarrow |f_1(z)| > |g_1(z)|$ *on* $|z| = 2$ $\Rightarrow$ $f_1(z)$ *and* $f_1(z) + g_1(z)| = P(z)$ *have the same number of zeros in* $|z| < 2$ *(by Rouche Theorem)* $\Rightarrow P(z)$ *has* 5 *zeros in* $|z| < 2$*, since* $f_1(z)$ *has* 5 *zeros in* $|z| < 2$*.*

Next let $f_2(z) = 16$ *and* $g_2(z) = z^5 - z$ *. Then, on* $|z| = 1$ *,* $|f_2(z)| = 16$ *and* $|g_2(z)| < 2$
$\Rightarrow |f_2(z)| > |g_2(z)|$ *on* $|z| = 1 \Rightarrow$ $f_2(z)$ *and* $f_2(z) + g_2(z)| = P(z)$ *have the same number of zeros in* $|z| < 1$ *(by Rouche Theorem)* $\Rightarrow P(z)$ *has no zeros in* $|z| < 1$*, since* $f_2(z)$ *has no zeros in* $|z| < 1$*.*

Therefore, the number of zeros of polynomial $P(z)$ *lying in the annulus* $1 < |z| < 2$ *are* 5 *.* ***(8 marks)***

15. Let Mobius Transformation $M(z)=\frac{z+1}{z}$ maps the circle with center $1+i$ and radius 3 onto a circle with center a and radius r. Then,

(i) $a=$.......................... (ii) $r=$..........................

Solution: *Observe that* $M(z)=\frac{1}{z}+1$. *The image of given circle* $|z-1-i|=3$ *under the mapping* $w=\frac{1}{z}$ *is*

$$\left|\frac{1}{w}-1-i\right|=3\Rightarrow |(1+i)w-1|^2=9|w|^2\Rightarrow 2|w|^2-2\operatorname{Re}((1+i)w)+1=9|w|^2$$

$\Rightarrow 7(u^2+v^2)+2u-2v-1=0,$ *where* $w=u+iv$

whose center is $-\frac{1}{7}+\frac{1}{7}i$, *radius is* $\sqrt{\frac{1}{49}+\frac{1}{49}+\frac{1}{7}}=\sqrt{\frac{9}{49}}=\frac{3}{7}$.

Therefore, the center and radius of the required circle is $a=(1-\frac{1}{7})+\frac{1}{7}i=\frac{6}{7}+\frac{1}{7}i$, $r=\frac{3}{7}$.

(4 marks for each correct answer, 2 marks out of 4 if the value of a ***is found as*** $-\frac{1}{7}+\frac{1}{7}i$***)***

16. The Mobius transformation $w=R(z)$ that maps points $z=1,\ i\ and\ -i$ to the points $w=1,\ i\ and\ -1+2i$, respectively, is expressed as $\frac{w-i}{w+1-2i}=\alpha\frac{z-i}{z+i}$, where α is a complex number. Then,

$(i)\,\alpha=$............ (ii) $R(-1)=$.............

Solution: *The Mobius transformation* $w=R(z)$ *is given by*

$$\frac{w-i}{w+1-2i}\Big/\frac{1-i}{2-2i}=\frac{z-i}{z+i}\Big/\frac{1-i}{1+i}\Rightarrow \alpha=\frac{1+i}{2(1-i)}=\frac{i}{2}.$$

$$\frac{w-i}{w+1-2i}\times 2=-1\ at\ z=-1\Rightarrow w=R(-1)=-\frac{1}{3}+\frac{4}{3}i$$

(4 marks for each correct answer)